Lecture Notes, October 23, 2012

The Arrow-Debreu Model of General Competitive Equilibrium

General Equilibrium Theory: Who was Prof. Debreu and why did he have his own parking space in Berkeley's Central Campus??

Nobel Prizes: Arrow, Debreu

What does mathematical general equilibrium theory do? Tries to put microeconomics on same basis of logical precision as algebra or geometry. Axiomatic method: allows generalization; clearly distinguishes assumptions from conclusions and clarifies the links between them.

Four ideas about writing an economic theory:

Ockam's razor (KISS - Keep it simple, stupid.), improves generality

Testable assumptions (logical positivism), avoids vacuity

Link with experience, robustness, Solow "All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive. A "crucial" assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic. When the results of a theory seem to flow specifically from a special crucial assumption, then if the assumption is dubious, the results are suspect. " (Contribution to the Theory of Economic Growth, 1956)

Precision, reliable results, Hugo Sonnenschein: "In 1954, referring to the first and second theorems of classical welfare economics, Gerard wrote 'The contents of both Theorems ... are old beliefs in economics. Arrow and Debreu have recently treated these questions with techniques permitting proofs.' This statement is precisely correct; once there were beliefs, now there was knowledge.

"But more was at stake. Great scholars change the way that we think about the world, and about what and who we are. The Arrow-Debreu model, as communicated in *Theory of Value* changed basic thinking, and it quickly became the standard model of price theory. It is the 'benchmark' model in Finance, International Trade, Public Finance, Transportation, and even macroeconomics. ... In rather short order it was no longer 'as it is' in Marshall, Hicks, and Samuelson; rather it became 'as it is' in *Theory of Value*." (remarks at the Debreu conference, Berkeley, 2005).

The Market, Commodities and Prices

N commodities

$x = (x_1, x_2, x_3, ..., x_N) \in \mathbf{R}^N$, a commodity bundle

The market takes place at a single instant, prior to the rest of economic activity.
commodity = good or service completely specified
description
location
date (of delivery)

Time: A futures market: no reopening of trade. This issue can be complex. We'll deal with it more thoroughly in Chapter 20.

Price system : $p = (p_1, p_2, ..., p_N) \neq 0$.
$p_i \geq 0$ for all i = 1, ..., N.
Value of a bundle $x \in R^N$ at prices p is $p\cdot x$.

Bounded and Unbounded Firm Technologies

Prices should communicate scarcity (and the boundedness of attainable outputs) to firms. Firms should be able to think: "If we had unbounded inputs we could produce unbounded outputs." So ideally we'd like a model where the firm could decide on arbitrarily large inputs and outputs --- then the price system would communicate that such a plan is unprofitable.

But a firm trying to plan a profit-maximizing production plan on an unbounded technology set may result in no well-defined plan. There may be no profit maximum since arbitrarily large plans may appear to produce arbitrarily large profits.

Modeling strategy:

1. Model production with bounded (and closed) firm technology, $\mathcal{Y}^j$. Then there will surely be a maximum profit achievable.

2. Demonstrate that with finite inputs and convex unbounded technology, Y^j, only finite outputs are possible.

3. Based on 2, consider the an artificial model economy with unbounded technology constrained to a bounded subset, $\tilde{Y}^j$, which then fulfills 1. Find market-clearing prices.

4. Show that the profit maximizing plan does not change when $\tilde{Y}^j$ is replaced by Y^j. Hence prices are still market-clearing. A mathematician's trick:

rearrange the problem to one you know how to solve, (reduce it to the previous --- already solved --- case).

Firms and Production Technology

F , j ∈ F , j = 1, ... , #F. Fixed finite number of firms.

Production technology: $\mathcal{Y}^j \subset R^N$. $y \in \mathcal{Y}^j$ (the script Y notation is to emphasize that $\mathcal{Y}^j$ is bounded).

Negative co-ordinates of y are inputs; positive co-ordinates are outputs.

$y \in \mathcal{Y}^j$, $y = (-2, -3, 0, 0, 1)$

This is a more general specification than a production function. The relationship is $f^j(x) \equiv \max \{ w \mid (-x, w) \in \mathcal{Y}^j\}$.

The Form of Production Technology

P.II. $0 \in \mathcal{Y}^j$.

P.III. $\mathcal{Y}^j$ is closed. (continuity)

P.VI $\mathcal{Y}^j$ is a bounded set for each j ∈ F. (We'll dispense with this evenutally)

P.III and P.VI ⇒ $\mathcal{Y}^j$ is compact

Compactness of $\mathcal{Y}^j$ is needed to be sure that profit maximization is well-defined, but P.VI is an ugly assumption: boundedness of a firm's attainable production possibilities should be communicated by the price system --- not by assumption. Chapter 15 of Starr's book weakens the assumption by showing that --- even when the firm's technology set is unbounded --- under weak assumptions, the set of attainable plans is bounded. Then circumscribe the unbounded technology set by a ball strictly containing the attainable plans. Apply the analysis of chaps. 11 - 14 to the artificially circumscribed production technology --- there will be an equilibrium (theorem 14.1) and an equilibrium is necessarily attainable, so the circumscibing ball is not a binding constraint in equilibrium. Then delete the artificial circumscribing ball; the prices and allocation remain an equilibrium. Conclusion: P.VI can be eliminated but it's a bit of work to do it.

Strictly Convex Production Technology

P.V. For each $j \in$ F, $\mathcal{Y}^j$ is strictly convex.

Convexity implies no scale economies, no indivisibilities.

$p \in R^N_+$, $p = (p_1, p_2, ..., p_N)$, $p \neq 0$.

$\tilde{S}^j(p) \equiv \{y^{*j} \mid y^{*j} \in \mathcal{Y}^j, \quad p \cdot y^{*j} \geq p \cdot y$ for all $y \in \mathcal{Y}^j\}$.

Theorem 11.1: Assume P.II, P.III, P.V, and P.VI. Let $p \in R^N_+, p \neq 0$. Then $\tilde{S}^j(p)$ is a well defined continuous point-valued function.

Proof:
Well defined: $\tilde{S}^j(p)$ = maximizer of a continuous real-valued function on a compact set.
Point-valued: Strict convexity of $\mathcal{Y}^j$, P.V. Point valued-ness implies that $\tilde{S}^j(p)$ is a function.
Continuity: Let $p^\nu \in R^N_+; \nu = 1, 2, \ldots; p^\nu \neq 0$, $p^\nu \to p^o \neq 0$. Show $\tilde{S}^j(p^\nu) \to \tilde{S}^j(p^o)$. Note: this is a consequence of the Maximum Theorem (see Berge, *Topological Spaces*), but we can provide a direct proof here, by contradiction. Suppose not. Then there is a cluster point of the sequence $\tilde{S}^j(p^\nu)$, y* so that $y^* \neq \tilde{S}(p^o)$ and $p^o \bullet \tilde{S}^j(p^o) > p^o \bullet y^*$ (why does this inequality hold? by definition of $\tilde{S}^j(p^o)$). That is there is a subsequence p^ν so that $\tilde{S}^j(p^\nu) \to y^*$. Note that $p^\nu \bullet \tilde{S}^j(p^o) \to p^o \cdot \tilde{S}^j(p^o)$. We have $p^\nu \cdot \tilde{S}^j(p^\nu) \to p^o \cdot y^*$ and $p^o \cdot \tilde{S}^j(p^o) > p^o \cdot y^*$. But the dot product is a continuous function of its arguments, so for ν large, $p^\nu \bullet \tilde{S}^j(p^o) > p^\nu \bullet \tilde{S}^j(p^\nu)$, a contradiction. This is a contradiction since $\tilde{S}^j(p^\nu)$ is the profit maximizer at p^ν. Thus, we must have $\tilde{S}^j(p^\nu) \to \tilde{S}^j(p^o)$. Q.E.D.

Lemma 1: (homogeneity of degree 0) Assume P.II, P.III, and P.VI. Let $\lambda > 0$, $p \in R^N_+$. Then $\tilde{S}^j(\lambda p) = \tilde{S}^j(p)$.

$$\tilde{S}(p) \equiv \sum_{j \in F} \tilde{S}^j(p)$$

4.4 Attainable Production Plans

Definition: A sum of sets $\mathcal{Y}^j$ in R^N, is defined as $\mathcal{Y} = \sum_j \mathcal{Y}^j$ is the set $\{y \mid y = \sum_j y^j \text{ for some } y^j \in \mathcal{Y}^j\}$.

Aggregate technology set:

$\mathcal{Y} \equiv \sum_{j \in F} \mathcal{Y}^j$.

Initial inputs to production $r \in R^N_+$

Definition: Let $y \in \mathcal{Y}$. Then y is said to be attainable if $y + r \geq 0$.

$y \in \mathcal{Y}$ is attainable if $(y + r) \in [\mathcal{Y} + \{r\}] \cap R^N_+$.

Note that under this definition, and P.II, P.III, P.V, P.VI the attainable set of outputs is compact and convex.